Consistent yield curve modelling

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joint work with

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Interest rate models

Challenge:

- Consistent recalibration of model parameters.

Classical approach: affine factor models for the short rate

- Main example: Vasiček model

$$dr(t) = \big(b + \beta r(t)\big)dt + \sigma dW(t).$$

- More generally, affine multi-factor models, possibly with jumps.

Calibration to initial yield curves

Calibration to initial yield curves

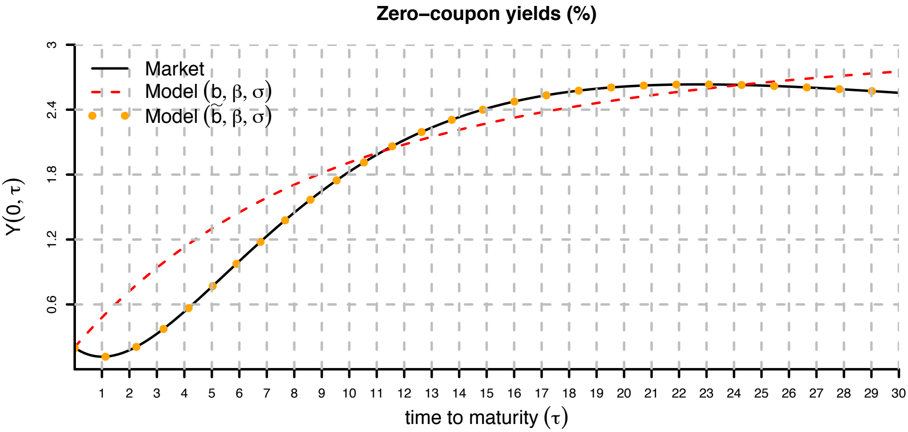
Problem

- Homogeneous models: No exact fit to market yield curves.
- Therefore, inconsistency between model and market.

Solution

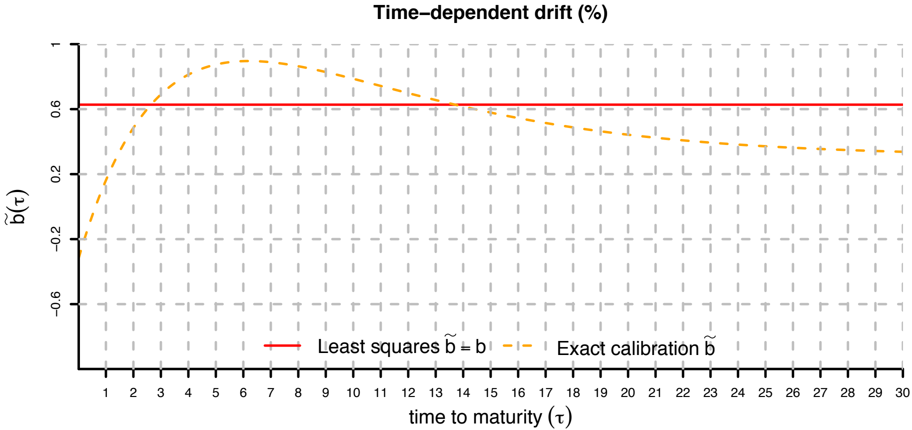
- Use Hull-White extensions.
- Obtain reformulations as HJM models.

Calibration to initial yield curves



Calibration of Hull-White extensions to initial yield curves.
(Vasiček model)

Hull-White extensions



Constant drift b versus time-dependent drift $\widetilde{b}(t)$.

Factor models as HJM models

HJM equation

- In Vasiček models with fixed parameters (β, σ), forward rates satisfy

$$df_t = \left(\frac{\partial}{\partial \tau} f_t + \mu^{\mathrm{HJM}}_{\beta,\sigma} \right) dt + \sigma^{\mathrm{HJM}}_{\beta,\sigma} dW_t,$$

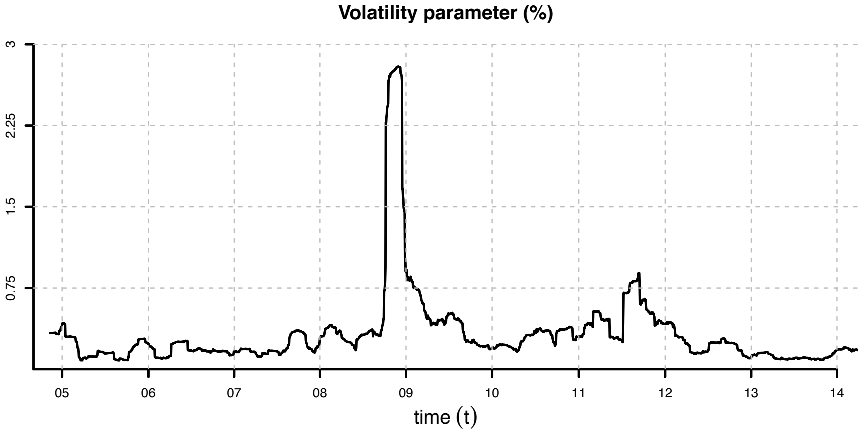
where each f_t is a curve of forward rates indexed by τ.

Properties

- Finite-dimensional realisation of the HJM equation.
- Easy to simulate.
- Calibration reduces to an estimation problem because of the analytical formulas for bond prices.

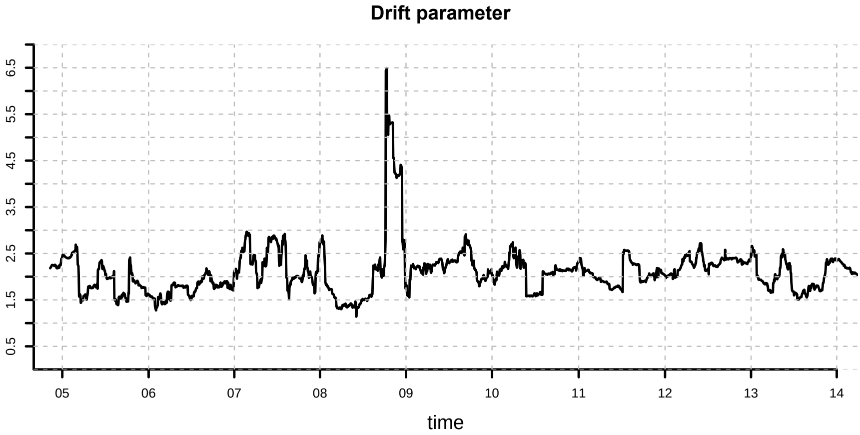
Time-varying parameters

Iterative recalibration of factor models



Time series of calibrated Vasiček volatilities σ
(AAA rated Euro area government bonds)

Iterative recalibration of factor models



Time series of calibrated Vasiček speeds of mean reversion $-\beta$
(AAA rated Euro area government bonds)

Time-varying parameters

Motivation

- Iterative recalibration results in time series of model parameters.
- The model should anticipate that parameters are subject to change.

Problem

- Introducing stochastic parameters in affine factor models destroys their good properties.

Solution (Consistent Recalibration Models)

- Make HJM parameters stochastic, but stick to HJM volatilities coming from affine factor models.

Consistent Recalibration Models

Consistent recalibration models

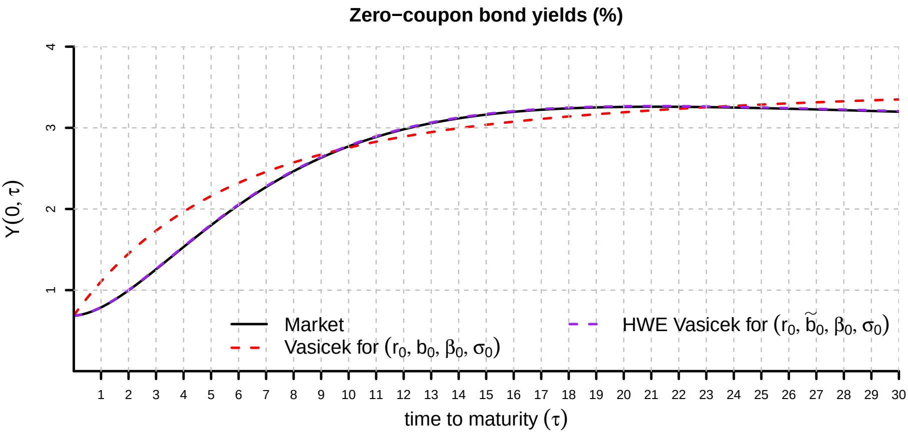
Definition

- In consistently recalibrated Vasiček models, forward rates satisfy

$$df_t = \left(\frac{\partial}{\partial\tau} f_t + \mu^{\mathrm{HJM}}_{\beta_t,\sigma_t}\right) dt + \sigma^{\mathrm{HJM}}_{\beta_t,\sigma_t} dW_t.$$

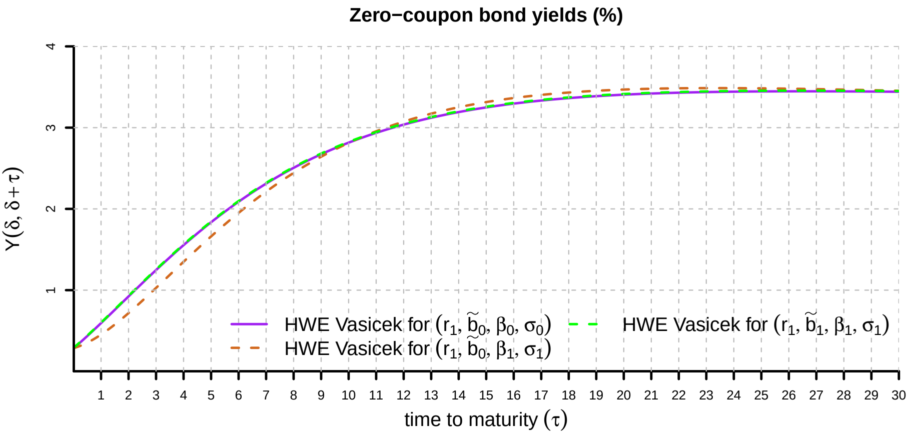
- Here, $\mu^{\mathrm{HJM}}_{\beta,\sigma}, \sigma^{\mathrm{HJM}}_{\beta,\sigma}$ denote the HJM drift and volatility of the Vasiček model with parameters β, σ.
- β_t, σ_t are stochastic processes.

Simulation



- Assume that (β_t, σ_t) is piecewise constant.
- Fix r_0 and parameters $\widetilde{b}_0, \beta_0, \sigma_0$ calibrated to the market.
- Simulate r_1 starting from r_0 using these parameters.

Simulation



- Choose new parameters β_1, σ_1.
- Calibrate $\widetilde{b}_1$ to the yield curve at $t = 1$ of the model with old parameters and repeat.

A semigroup perspective

The simulation algorithm as a splitting scheme

- Assume that the parameter process (β_t, σ_t) is Markovian.
- Then the simulation scheme with piecewise constant parameter process (β_t, σ_t) is an *exponential Euler splitting schemes* for the joint evolution of (f_t, β_t, σ_t).

Convergence of the simulation scheme

- By semigroup methods, one obtains convergence to solutions of

$$df_t = \left(\frac{\partial}{\partial \tau} f_t + \mu^{\mathrm{HJM}}_{\beta_t, \sigma_t}\right) dt + \sigma^{\mathrm{HJM}}_{\beta_t, \sigma_t} dW_t.$$

A geometric perspective

Foliations of the space of forward rate curves

- Each choice of (β, σ) corresponds to a foliation of the space of forward rate curves.
- In factor models, forward rate curves evolve on single leaves of the foliation.
- In CRC models, forward rate evolutions are tangent to the foliation corresponding to (β_t, σ_t), at all t.

An empirical perspective

Realised covariations

- Consider the 10×10 matrix of realised covariations (on time-windows $[t, t+1]$ of one year) between yields of maturities $\tau_i, \tau_j \in \{1, \ldots, 10\}$.
- On the Euro-area government bond market, this matrix had ranks 7–10 over the years 2005–2013.
- In the Vasiček and CIR model, this matrix has rank 1.
- In Vasiček CRC models, where β is updated stochastically every week, this matrix has rank 7–9, in our simulations.
- In the continuous-time limit of the model, the matrix has full rank.

Conclusion

Advantages of HJM models...

- Exact fits to initial yield curves can be achieved.
- Dynamics can be specified independently of the initial fit.
- Time-dependent parameters pose no problem.
- No arbitrage.

...combined with advantages of factor models

- Simulation is easy.
- Analytical bond pricing formulas hold.
- Calibration reduces to an estimation problem.

Thank you

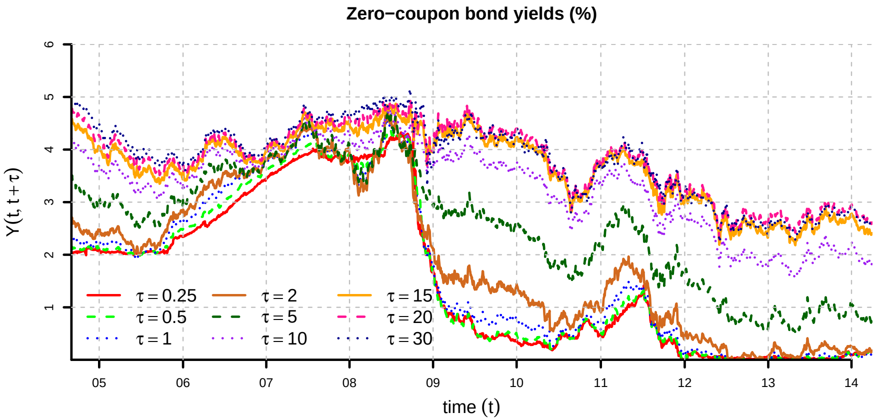
Literature

[1] Anja Richter and Josef Teichmann. "Discrete Time Term Structure Theory and Consistent Recalibration Models". In: *arXiv preprint arXiv:1409.1830* (2014).

[2] Philipp Dörsek and Josef Teichmann. "Efficient simulation and calibration of general HJM models by splitting schemes". In: *SIAM Journal on Financial Mathematics* 4.1 (2013), pp. 575–598.

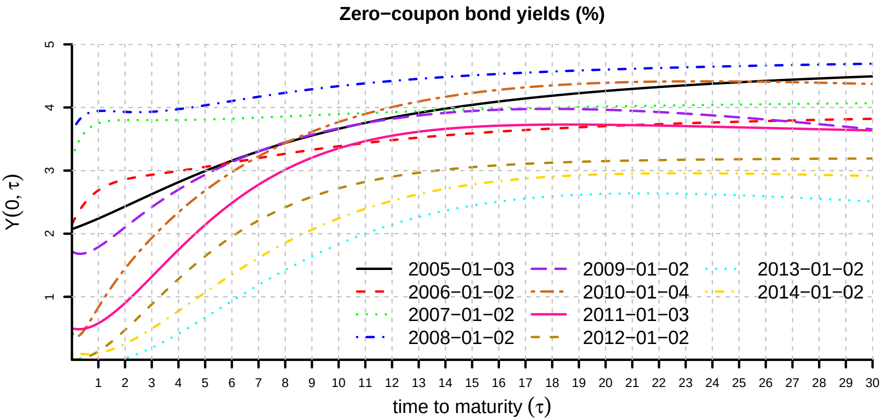
[3] Jan Kallsen and Paul Krühner. "On a Heath-Jarrow-Morton approach for stock options". In: *arXiv preprint arXiv:1305.5621* (2013).

Term structure of interest rates



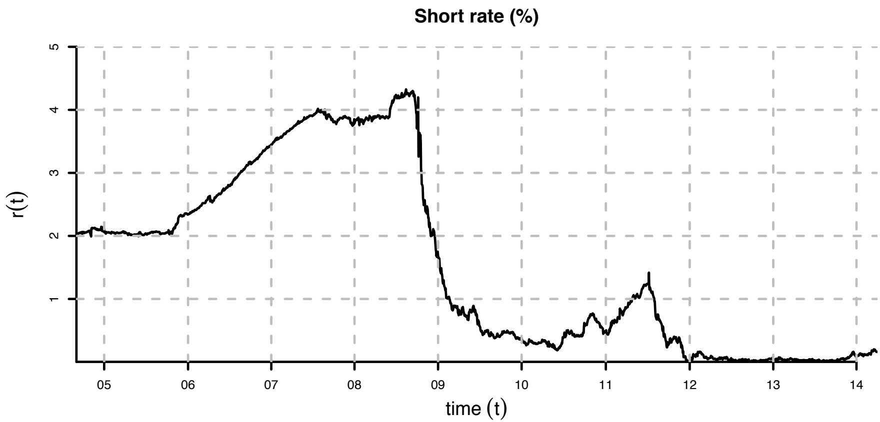
Stochastic evolution of yields with fixed maturities

Term structure of interest rates



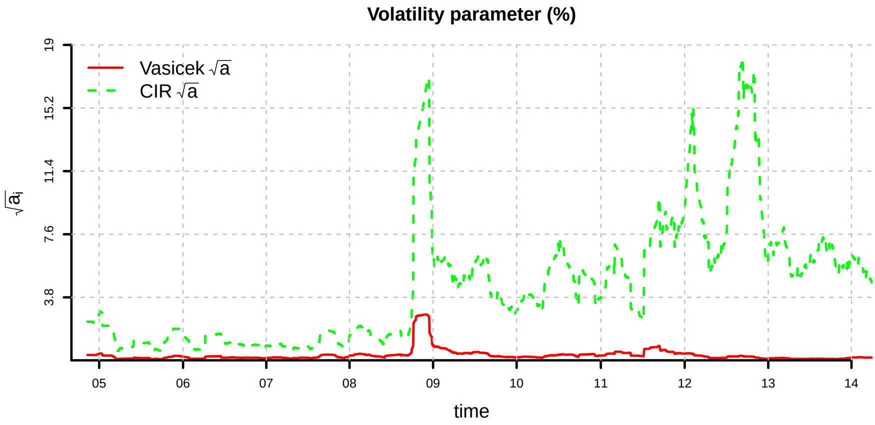
Term structure of yields on a fixed day

Short-rate along time series



The short rate, as obtained by approximation by 3-month yields.

Instantaneous volatility σ of the short rate

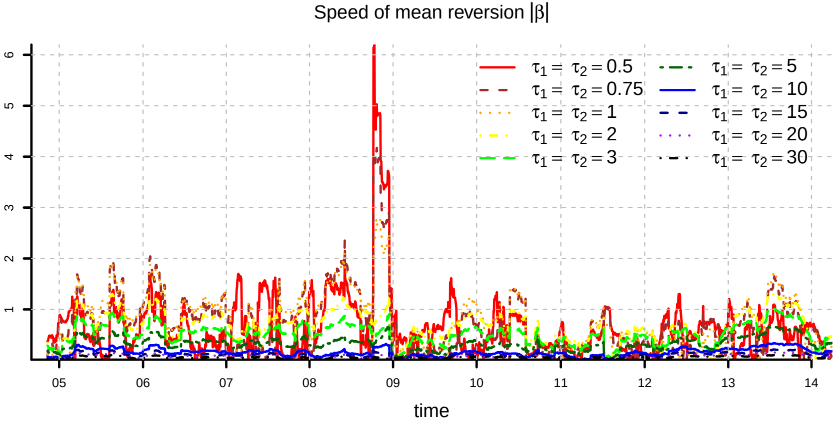


The instantaneous volatility $\sigma = \sqrt{a}$ of the short rate, obtained by path-wise estimation.

Vasiček mean reversion parameter β

The mean reversion parameter $\beta < 0$, estimated from the covariation of yields.

Vasiček mean reversion parameter β



The estimated β depends on the times to maturity of the yields used in the estimation.